A Modified Similarity Degree for C*-algebras

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Abstract. We define variants of Pisier's similarity degree for unital C*-algebras and use direct integral theory to obtain new results. We prove that if every II_1 factor representation of a separable C*-algebra $\mathcal{A}$ has property Γ, then the similarity degree of $\mathcal{A}$ is at most 11.

G. Pisier's *similarity degree* [**10**]-[**14**] has been one of the most far-reaching advances on R. Kadison's similarity problem [**7**], which asks whether every bounded homomorphism ρ from a C*-algebra $\mathcal{A}$ into the operators on a Hilbert space must be similar to a $*$-homomorphism. Many classical results on the similarity degree are contained in [**14**]. There has been some recent interest on this subject [**16**], [**8**], [**6**], [**9**], [**4**].

In this paper we define two variants of G. Pisier's similarity degree [**10**]-[**14**], one for C*-algebras and one for von Neumann algebras. Our main result (Theorem 8) relates our C*-invariant for a separable unital C*-algebra to the supremum of the W*-invariant of all the II_1 factor representations of the algebra. This result yields bounds on the similarity degree in some new cases, including some crossed products and the class of separable unital C*-algebras whose II_1 factor representations all have property Γ.

It was shown by U. Haagerup [**2**] (see also the union of [**3**] and [**18**]) that a bounded homomorphism ρ on a C*-algebra is similar to a $*$-homomorphism if and only if it is completely bounded, i.e., $\|\rho\|_{cb} = \sup_{n\in\mathbb{N}} \|\rho_n\| < \infty$, where ρ_n is defined on $\mathcal{M}_n(\mathcal{A})$ by

$$\rho_n([a_{ij}]) = [\rho(a_{ij})]$$

for every $[a_{ij}] \in \mathcal{M}_n(\mathcal{A})$.

G. Pisier defined the *similarity degree* of $\mathcal{A}$, denoted by $d(\mathcal{A})$ to be the smallest positive integer d (if one exists) for which there is a positive number κ such that, for every unital C*-algebra $\mathcal{B}$ and every bounded unital homomorphism $\rho : \mathcal{A} \to \mathcal{B}$ we have

$$\|\rho\|_{cb} \leq \kappa \|\rho\|^{d(\mathcal{A})}.$$

If there is no such pair d, κ, we define $d(\mathcal{A}) = \infty$. We denote the smallest κ by $\kappa(\mathcal{A})$.

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There has been much attention focused on the similarity degree without much attention to κ. It is known [**14**] that when $\mathcal{A}$ is finite-dimensional, $d(\mathcal{A}) = 1$ and that when $\mathcal{A}$ is infinite-dimensional, then $d(\mathcal{A}) \geq 2$. It follows, for example that, for any strictly increasing sequence $\{\mathcal{A}_n\}$ of finite-dimensional C*-algebras, $d(\mathcal{A}_n) = 1$ for every $n \in \mathbb{N}$, but $\kappa(\mathcal{A}_n) \to \infty$, otherwise there would be an infinite-dimensional direct limit $\mathcal{A}$ with $d(\mathcal{A}) = 1$. Hence determining the similarity degree of a direct sum or direct limit of C*-algebras with the same similarity degree is difficult, without also controlling the κ's. This motivates us to introduce a new invariant that incorporates both d and κ.

Definition 1. *We define the* modified similarity degree $\hat{d}(\mathcal{A})$ *of $\mathcal{A}$ to be the smallest positive number $\gamma \geq d(\mathcal{A})$ such that, for every bounded unital algebra homomorphism $\rho : \mathcal{A} \to B(H)$, we have*

$$\|\rho\|_{cb} \leq \gamma \|\rho\|^{\gamma}.$$

Definition 2. *If $\mathcal{A}$ is a von Neumann algebra, we similarly define $\hat{d}_*(\mathcal{A})$ to be the smallest positive number γ such that*

$$\|\rho\|_{cb} \leq \gamma \|\rho\|^{\gamma}$$

*whenever a bounded unital homomorphism $\rho : \mathcal{A} \to B(H)$ is ultra*strong-ultra*strong continuous on the closed unit ball of $\mathcal{A}$, equivalently, ultrastrong-ultrastrong continuous on the closed unit ball of $\mathcal{A}^{sa} = \{\operatorname{Re} a : a \in \mathcal{A}\}$.*

The following result is elementary.

Lemma 3. *Suppose $\mathcal{A}$ is a unital C*-algebra. Then*

(1) *If $\mathcal{J}$ is a closed $*$-ideal in $\mathcal{A}$, then*

$$\hat{d}(\mathcal{A}/\mathcal{J}) \leq \hat{d}(\mathcal{A})$$

(2)

$$d(\mathcal{A}) \leq \hat{d}(\mathcal{A}) \leq \max(d(\mathcal{A}), \kappa(\mathcal{A})).$$

(3) *If $\mathcal{A}$ is the norm closure of the union of an increasingly directed family $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ of unital C*-algebras, then*

$$\hat{d}(\mathcal{A}) \leq \liminf_{\lambda} \hat{d}(\mathcal{A}_\lambda).$$

(4) *If, in statement (3) above, $\mathcal{A}$ is the weak operator closure (or, strong operator closure) of the union of the $\mathcal{A}_\lambda$'s, then*

$$\hat{d}_*(\mathcal{A}) \leq \liminf_{\lambda} \hat{d}_*(\mathcal{A}_\lambda'') \leq \liminf_{\lambda} \hat{d}(\mathcal{A}_\lambda).$$

Proof. (1) and (2) are obvious.

(3). If $\rho : \mathcal{A} \to B(H)$ is a bounded unital homomorphism, then, $\forall\ n \in \mathbb{N}$,

$$\begin{aligned}
\|\rho_n\| = \lim_{\lambda} \|\rho_n|_{\mathcal{A}_\lambda}\| &\leq \lim_{\lambda} \|\rho_n|_{\mathcal{A}_\lambda}\|_{cb} \\
&\leq \liminf_{\lambda} \hat{d}\left(\mathcal{A}_\lambda\right) \|\rho|_{\mathcal{A}_\lambda}\|^{\hat{d}(\mathcal{A}_\lambda)} \\
&\leq \liminf_{\lambda} \hat{d}\left(\mathcal{A}_\lambda\right) \|\rho\|^{\hat{d}(\mathcal{A}_\lambda)} \\
&\leq \left(\liminf_{\lambda} \hat{d}\left(\mathcal{A}_\lambda\right)\right) \|\rho\|^{\liminf_\lambda \hat{d}(\mathcal{A}_\lambda)}.
\end{aligned}$$

Since $\|\rho\|_{cb} = \sup_{n\in\mathbb{N}} \|\rho_n\|$, the desired result is proved.

(4). Let $\mathcal{B}$ be the norm closure of the union of the $\mathcal{A}_\lambda$'s. If $\rho : \mathcal{A} \to B\left(H\right)$ is a unital homomorphism that is ultrastrong-ultrastrong continuous on the closed unit ball of $\mathcal{A}^{sa}$, then it follows from the Kaplansky density theorem that $\|\rho\| = \|\rho|_{\mathcal{B}}\|$ and $\|\rho\|_{cb} = \|\rho|_{\mathcal{B}}\|_{cb}$. The rest follows from (3). □

Corollary 4. *Suppose $\mathcal{M}$ is a von Neumann Algebra. Then*

$$\hat{d}_*\left(\mathcal{M}\right) \leq \inf\left\{\hat{d}\left(\mathcal{A}\right) : \mathcal{A} \subseteq \mathcal{M},\ \mathcal{A} \text{ a } C^*\text{-algebra, } \mathcal{A}'' = \mathcal{M}\right\}.$$

It was shown by U. Haagerup [**2**] that if a unital C*-algebra has no tracial states, then $d\left(\mathcal{A}\right) = 3$ with $\kappa = 1$, which implies $\hat{d}\left(\mathcal{A}\right) \leq 3$. Hence if $\mathcal{M}$ is a type I_∞, type II_∞ or type III factor, then $d\left(\mathcal{M}\right) = \hat{d}\left(\mathcal{M}\right) = 3$. In particular, $\hat{d}\left(\mathcal{B}\left(\ell^2\right)\right) = 3$. We see that $\hat{d}_*$ does a little better.

Corollary 5. *If $\mathcal{M}$ is a hyperfinite von Neumann algebra, then $\hat{d}_*\left(\mathcal{M}\right) \leq 2$.*

Corollary 6. *Suppose $\mathcal{A}$ is a unital C*-algebra. Then*

$$\hat{d}\left(\mathcal{A}\right) = \hat{d}_*\left(\mathcal{A}^{\#\#}\right).$$

If τ is a tracial state on a unital C*-algebra $\mathcal{A}$, we let $L^2\left(\mathcal{A},\tau\right)$ denote the Hilbert space induced by the inner product $\langle a,b\rangle = \tau\left(b^*a\right)$ and let $\pi_\tau : \mathcal{A} \to B\left(L^2\left(\mathcal{A},\tau\right)\right)$ be the GNS representation on $\mathcal{A}$ defined by

$$\pi_\tau\left(a\right)\left(b\right) = ab$$

whenever $a, b \in \mathcal{A}$. We define

$$\mathcal{M}_\tau\left(\mathcal{A}\right) = \pi_\tau\left(\mathcal{A}\right)''$$

be the von Neumann algebra generated by $\pi_\tau\left(\mathcal{A}\right)$. It is known (e.g., see [**5**]) that τ is an extreme point of the set of tracial states on $\mathcal{A}$ if and only if $\mathcal{M}_\tau\left(\mathcal{A}\right)$ is a finite factor von Neumann algebra, and, in this case we call τ a *factor tracial state.*

Definition 7. *We define the* modified tracial similarity degree *of a unital C*-algebra $\mathcal{A}$ as*

$$\hat{d}_{tr}\left(\mathcal{A}\right) = \sup\left\{\hat{d}_*\left(\mathcal{M}_\tau\left(\mathcal{A}\right)\right) : \tau \text{ is a factor tracial state of } \mathcal{A}\right\}.$$

Our main result explicitly shows how finding $\hat{d}$ for separable C*-algebras reduces to finding $\hat{d}_*$ for II_1 factor von Neumann algebras.

THEOREM 8. *Suppose $\mathcal{A}$ is a separable unital C*-algebra. Then,*

$$d(\mathcal{A}) \leq \hat{d}(\mathcal{A}) \leq 2 + 3\max\left(3, \hat{d}_{tr}(\mathcal{A})\right).$$

In particular, for every unital bounded homomorphism $\rho : \mathcal{A} \to B(\ell^2)$

$$\|\rho\|_{cb} \leq \max\left(3, \hat{d}_{tr}(\mathcal{A})\right) \|\rho\|^{2+3\max\left(3,\hat{d}_{tr}(\mathcal{A})\right)}.$$

PROOF. Suppose $\rho : \mathcal{A} \to B(H)$ is a unital faithful $*$-homomorphism where H is a separable infinite-dimensional Hilbert space. By replacing ρ with $\rho^{(\infty)} = \rho \oplus \rho \oplus \cdots$, we can assume that ρ is unitarily equivalent to $\rho \oplus \rho \oplus \cdots$. We can extend ρ to a normal unital homomorphism $\hat{\rho} : \mathcal{A}^{\#\#} \to B(H)$ such that $\hat{\rho}|_{\mathcal{A}} = \rho$ and such that $\hat{\rho}$ is unitarily equivalent to $\hat{\rho}^{(\infty)}$. Since $\ker \hat{\rho}$ is a weak*-closed ideal in the von Neumann algebra $\mathcal{A}^{\#\#}$, there is a central projection $Q \in \mathcal{A}^{\#\#}$ such that $\ker \hat{\rho} = (1-Q)\mathcal{A}^{\#\#}$. Let $\mathcal{M} = Q\mathcal{A}^{\#\#}$. Since $\hat{\rho}$ is normal, we see that $\hat{\rho}(\mathcal{M})$ is weak*-closed (Krein-Shmulyan) and $\sigma = \hat{\rho}|_{\mathcal{M}} : \mathcal{M} \to \hat{\rho}(\mathcal{M})$ is a weak*-weak* homeomorphism. Thus the predual of $\mathcal{M}$ is separable, so there is a normal faithful $*$-homomorphism $\pi : \mathcal{M} \to B(H)$, and we can assume that π is unitarily equivalent to $\pi^{(\infty)}$. Hence we can assume that $\mathcal{A} \subseteq \mathcal{M} \subseteq B(H)$, $\mathcal{M} = \mathcal{A}''$, $id_{\mathcal{M}}$ is unitarily equivalent to $id_{\mathcal{M}}^{(\infty)}$ and that $\sigma : \mathcal{M} \to B(H)$ is a faithful normal homomorphism such that $\sigma|_{\mathcal{A}} = \rho$ and σ is unitarily equivalent to $\sigma^{(\infty)}$. It follows from the Pisier-Ringrose theorem [**10**], [**17**] that σ is ultra*strong-ultra*strong continuous. Since ρ and π have infinite multiplicity, we know that σ is SOT-SOT continuous on $\mathcal{M}^{sa}$.

It follows from direct integral theory that, up to unitary equivalence, there is a separable infinite-dimensional Hilbert space K and a probability measure space (Ω, μ) such that $H = L^2(\mu, K)$ and the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ is

$$\{\varphi \in L^\infty(\mu, B(H)) : \varphi(\omega) \in \mathbb{C}1 \text{ a.e.}\}.$$

Hence $\mathcal{M} \subseteq \mathcal{Z}(\mathcal{M})' = L^\infty(\mu, B(H))$ and if $\{\varphi_1, \varphi_2, \ldots\}$ is a selfadjoint norm-dense subset of the closed unit ball of $\mathcal{A}^{sa}$, and hence a strong-operator-dense subset of the closed unit ball of $\mathcal{M}^{sa}$, and, for each $\omega \in \Omega$, we let $\mathcal{M}_\omega = \{\varphi_1(\omega), \varphi_2(\omega), \ldots\}''$, we have

$$\mathcal{M} = \int_\Omega^\oplus \mathcal{M}_\omega d\mu(\omega),$$

and $\mathcal{M}$ is $\{\varphi \in L^\infty(\mu, B(H)) : \varphi(\omega) \in \mathcal{M}_\omega \ a.e\}$. Moreover, we can further assume that $\{\varphi_1(\omega), \varphi_2(\omega), \ldots\}$ is strong-operator-dense in the closed unit ball of $\mathcal{M}_\omega^{sa}$ a.e., and that there is a measurable family $\{\pi_\omega : \omega \in \Omega\}$ of unital $*$-homomorphisms from $\mathcal{A}$ to $B(K)$ such that,

$$id_{\mathcal{A}} = \int_\Omega^\oplus \pi_\omega d\mu$$

and

$$\pi_\omega(\mathcal{A})'' = \mathcal{M}_\omega \text{ a.e. }.$$

Now we want to look at the restriction of σ to $\mathcal{Z}(\mathcal{M})$, which is a bounded unital normal injective homomorphism with $\|\rho\| = \|\sigma\|$. Since $d(\mathcal{Z}(\mathcal{M})) = 2$, there is an invertible operator $S \in B(H)$ such that $\|S\| \|S^{-1}\| \leq \|\rho\|^2$ and such that $\sigma_1(\cdot) = S\sigma(\cdot) S^{-1}$ is an injective normal $*$-homomorphism on $\mathcal{Z}(\mathcal{M})$. Hence

$$\|\sigma_1\| \leq \|\rho\|^2 \|\sigma\| = \|\rho\|^3.$$

Moreover, $\sigma(\cdot) = S\sigma_1(\cdot) S^{-1}$, so

$$\|\rho\|_{cb} = \|\sigma\|_{cb} \leq \|\rho\|^2 \|\sigma_1\|_{cb}.$$

Since σ_1 is unitarily equivalent to $\sigma_1^{(\infty)}$ and $id_{\mathcal{M}}$ is unitarily equivalent to $(id_{\mathcal{M}})^{(\infty)}$, we see that $\sigma_1|\mathcal{Z}(\mathcal{M})$ is unitarily equivalent to $id_{\mathcal{Z}(\mathcal{M})}$. By putting this unitary with S, we can assume that $\sigma_1(T) = T$ for every $T \in \mathcal{Z}(\mathcal{M})$. This means that σ_1 is an $L^\infty(\mu) = \mathcal{Z}(\mathcal{M})$ module homomorphism.

Since $\mathcal{M} \subseteq \mathcal{Z}(\mathcal{M})'$, we know that

$$\sigma_1(\mathcal{M}) \subseteq \mathcal{Z}(\mathcal{M})' = L^\infty(\mu, B(K)).$$

Hence we can find functions $\psi_1, \psi_2, \ldots \in L^\infty(\mu, B(K))$ such that

$$\sigma_1(\varphi_n) = \psi_n$$

for every $n \geq 1$.

It is important to note that the φ_n's and ψ_n's are actually representatives of equivalence classes since we identify functions that agree almost everywhere. So we must now take some care with sets of measure 0. First note that if $p(t_1, \ldots, t_n)$ is a $*$-polynomial, then

$$\sigma_1(p(\varphi_1, \ldots, \varphi_n)) = p(\psi_1, \ldots, \psi_n),$$

and

$$\|p(\psi_1, \ldots, \psi_n)\|_\infty \leq \|\sigma_1\| \|p(\varphi_1, \ldots, \varphi_n)\|_\infty.$$

We want to get a better estimate. For each $k \geq 1$, let E_k be the set of all $\omega \in \Omega$ such that

$$\|p(\psi_1(\omega), \ldots, \psi_n(\omega))\| > \|\sigma_1\| \|p(\varphi_1(\omega), \ldots, \varphi_n(\omega))\| + \frac{1}{k}.$$

Since σ_1 is an $L^\infty(\mu) = \mathcal{Z}(\mathcal{M})$ module homomorphism, we have

$$\sigma_1(\chi_{E_k} p(\varphi_1, \ldots, \varphi_n)) = \chi_{E_k} p(\psi_1, \ldots, \psi_n),$$

and

$$\|\chi_{E_k} p(\psi_1, \ldots, \psi_n)\|_\infty \leq \|\sigma_1\| \|\chi_{E_k} p(\varphi_1, \ldots, \varphi_n)\|_\infty.$$

It follows that $\mu(E_k) = 0$. Hence $\mu(\cup_{n\geq 1} \mu(E_k)) = 0$. Therefore

$$\|p(\psi_1(\omega), \ldots, \psi_n(\omega))\| \leq \|\sigma_1\| \|p(\varphi_1(\omega), \ldots, \varphi_n(\omega))\| \ \ a.e.$$

Now we have the following claim.

Claim 8.1: Let $\mathcal{P}$ denote the set of a $*$-polynomials with coefficients in $\mathbb{C}_\mathbb{Q} = \mathbb{Q} + i\mathbb{Q}$. We then have

$$\|p(\psi_1(\omega), \ldots, \psi_n(\omega))\| \leq \|\sigma_1\| \|p(\varphi_1(\omega), \ldots, \varphi_n(\omega))\| \ \ a.e.$$

for every $p \in \mathcal{P}$. By removing a set of measure zero from Ω, we can assume that the above relation hold for every $\omega \in \Omega$.

Let $\{1 = \gamma_1, \gamma_2, \ldots\}$ be an orthonormal basis for $L^2(\mu)$ and let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for K. Define $u_{i,k} \in H$ by

$$u_{i,k}(\omega) = \gamma_i(\omega) e_k, \qquad \forall\, i, k \in \mathbb{N}.$$

It is well known that $\{u_{i,k} : i, k \in \mathbb{N}\}$ is an orthonormal basis for $H = L^2(\mu, K)$. We can define a metric d_H on $B(H)$ by

$$d_H(S,T)^2 = \sum_{i,k=1}^{\infty} \frac{\|(S-T)u_{i,k}\|^2}{2^{i+k}}, \quad \forall\, S, T \in B(H)$$

and a metric d_K on $B(K)$ by

$$d_K(S,T)^2 = \sum_{k=1}^{\infty} \frac{\|(S-T)e_k\|^2}{2^k}, \quad \forall\, S, T \in B(K).$$

On bounded subsets of $B(H)$ (respectively, $B(K)$) the metric d_H (respectively, d_K) induces the strong operator topology.

We know that $\sigma : \mathcal{M} \to B(H)$ is ultrastrongly-ultrastrongly continuous on $\mathcal{M}^{sa}$. It follows that σ_1 is uniformly continuous on the closed unit ball $\mathcal{B}$ of $\mathcal{M}^{sa}$, since if $\{S_n\}, \{T_n\}$ are sequences in $\mathcal{B}$, we know $d_H(S_n, T_n) \to 0$ if and only if $S_n - T_n \to 0$ ultrastrongly, implying $\sigma_1(S_n - T_n) = \sigma_1(S_n) - \sigma_1(T_n) \to 0$ ultrastrongly, which implies $d_H(S_n, T_n) \to 0$. Therefore, we have the following claim.

Claim 8.2: Suppose $s, r \in \mathbb{N}$. Then there is a $t_{r,s} \in \mathbb{N}$ such that, for every $S, T \in \mathcal{B}$, we have

$$d_H(S,T) < \frac{1}{t_{r,s}} \Rightarrow d_H(\sigma_1(S), \sigma_1(T)) < \frac{1}{rs}.$$

Let $\mathcal{P}_1$ be the set of $p \in \mathcal{P}$ such that $\|p(\psi_1, \psi_2, \ldots)\| \le 1$ and $p(\psi_1, \psi_2, \ldots) = p(\psi_1, \psi_2, \ldots)^*$, and write

$$\mathcal{P}_1 \times \mathcal{P}_1 = \{(p_1, q_1), (p_2, q_2), \ldots\}.$$

Let r, s be in $\mathbb{N}$ and $t_{r,s}$ be as in Claim 8.2. Now suppose $j \in \mathbb{N}$, and let $E_{j,r,s}$ denote the set of all $\omega \in \Omega$ such that

$$d_K(p_j(\psi_1(\omega), \ldots, \psi_n(\omega)), q_j(\psi_1(\omega), \ldots, \psi_n(\omega)))^2 < \frac{1}{t_{r,s}},$$

and

$$d_K(p_j(\varphi_1(\omega), \ldots, \varphi_n(\omega)), q_j(\varphi_1(\omega), \ldots, \varphi_n(\omega)))^2 \ge \frac{1}{s}.$$

Let $F_{1,r,s} = E_{1,r,s}$ and let $F_{j+1,r,s} = E_{j+1,r,s} \setminus \cup_{1\leq i\leq j} E_{i,r,s}$ for $j \in \mathbb{N}$. Hence if $S = \sum_{j=1}^{\infty} \chi_{F_{j,r,s}} p_j(\psi_1, \ldots, \psi_n)$ and $T = \sum_{j=1}^{\infty} \chi_{F_{j,r,s}} q_j(\psi_1, \ldots, \psi_n)$, we have, for $i, k \in \mathbb{N}$, that

$$\begin{aligned}
&\|(S-T)\, u_{i,k}\|^2 \\
&\quad = \sum_{j=1}^{\infty} \int_{F_{j,r,s}} |\gamma_i(\omega)|^2 \,\|(p_j(\psi_1(\omega), \ldots, \psi_n(\omega)) - q_j(\psi_1(\omega), \ldots, \psi_n(\omega)))\, e_k\|^2 \, d\mu
\end{aligned}$$

and

$$\begin{aligned}
d_H(S,T)^2 &= \sum_{i,k=1}^{\infty} \frac{\|(S-T)\, u_{i,k}\|^2}{2^{i+k}} \\
&= \sum_{i,j=1}^{\infty} \int_{F_{j,r,s}} \left(\sum_{k=1}^{\infty} \frac{|\gamma_i(\omega)|^2 \,\|(p_j(\psi_1(\omega), \ldots, \psi_n(\omega)) - q_j(\psi_1(\omega), \ldots, \psi_n(\omega)))\, e_k\|^2}{2^{i+k}} \right) d\mu \\
&< \sum_{i=1}^{\infty} \frac{1}{t_{r,s}} \frac{1}{2^i} \int_{\Omega} |\gamma_i(\omega)|^2 \, d\mu = \frac{1}{t_{r,s}}.
\end{aligned}$$

Thus, by Claim 8.2, we have $d_H(\sigma_1(S), \sigma_1(T)) < \frac{1}{rs}$. However, since $\gamma_1(\omega) = 1$, we see that

$$\begin{aligned}
&\|(\sigma_1(S) - \sigma_1(T))\, u_{1,k}\|^2 = \\
&\qquad \sum_{j=1}^{\infty} \int_{F_{j,r,s}} \|(p_j(\varphi_1(\omega), \ldots, \varphi_n(\omega)) - q_j(\psi_1(\omega), \ldots, \psi_n(\omega)))\, e_k\|^2 \, d\mu,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{rs} &> d_H(\sigma_1(S), \sigma_1(T)) \\
&\geq \sum_{k=1}^{\infty} \frac{\|(\sigma_1(S) - \sigma_1(T))\, u_{1,k}\|^2}{2^{1+k}} \\
&\geq \frac{1}{2} \sum_{j=1}^{\infty} \int_{F_{j,r,s}} \sum_{k=1}^{\infty} \frac{\|(p_j(\varphi_1(\omega), \ldots, \varphi_n(\omega)) - q_j(\psi_1(\omega), \ldots, \psi_n(\omega)))\, e_k\|^2}{2^k} d\mu \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \int_{F_{j,r,s}} d_K(p_j(\varphi_1(\omega), \ldots, \varphi_n(\omega)), q_j(\psi_1(\omega), \ldots, \psi_n(\omega)))^2 \, d\mu \\
&\geq \frac{1}{2} \sum_{j=1}^{\infty} \int_{F_{j,r,s}} \frac{1}{s} d\mu = \frac{1}{2s} \mu\left(\cup_{j=1}^{\infty} F_{j,r,s}\right)
\end{aligned}$$

Hence

$$\frac{2}{r} > \mu\left(\cup_{j=1}^{\infty} F_{j,r,s}\right) = \mu\left(\cup_{j=1}^{\infty} E_{j,r,s}\right).$$

Let $W_{r,s} = \cup_{j=1}^{\infty} E_{j,r,s}$. Then $W_{r,s}$ is precisely the set of all $\omega \in \Omega$ for which there is a $(p,q) \in \mathcal{P}_1 \times \mathcal{P}_1$ such that

$$d_K \left(p\left(\psi_1\left(\omega\right), \ldots, \psi_n\left(\omega\right)\right), q\left(\psi_1\left(\omega\right), \ldots, \psi_n\left(\omega\right)\right)\right)^2 < \frac{1}{t_{r,s}}$$

and

$$d_K \left(p_j\left(\varphi_1\left(\omega\right), \ldots, \varphi_n\left(\omega\right)\right), q_j\left(\varphi_1\left(\omega\right), \ldots, \varphi_n\left(\omega\right)\right)\right)^2 \geq \frac{1}{s}.$$

It follows that

$$\mu\left(\cup_{s=1}^{\infty} \cap_{r=1}^{\infty} W_{r,s}\right) = 0.$$

Hence, if we throw away another set of measure 0, we can assume that

$$\cup_{s=1}^{\infty} \cap_{r=1}^{\infty} W_{r,s} = \emptyset,$$

which implies that

$$\cap_{s=1}^{\infty} \cup_{r=1}^{\infty} \left(\Omega \backslash W_{r,s}\right) = \Omega.$$

This means that

Claim 8.3: for every $\omega \in \Omega$ *and every* $s \in \mathbb{N}$ *there are an* $r \in \mathbb{N}$ *and, thus, a* $t_{r,s} \in \mathbb{N}$ *(as in Claim 8.2) such that* $\omega \notin W_{r,s}$*, i.e., for every* $(p,q) \in \mathcal{P}_1 \times \mathcal{P}_1$,

$$d_K \left(p\left(\psi_1\left(\omega\right), \ldots, \psi_n\left(\omega\right)\right), q\left(\psi_1\left(\omega\right), \ldots, \psi_n\left(\omega\right)\right)\right)^2 < \frac{1}{t_{r,s}}$$

implies that

$$d_K \left(p\left(\psi_1\left(\omega\right), \ldots, \psi_n\left(\omega\right)\right), q\left(\psi_1\left(\omega\right), \ldots, \psi_n\left(\omega\right)\right)\right)^2 < \frac{1}{s}.$$

It follows from Claim 8.1 that, for each $\omega \in \Omega$ and each $p \in \mathcal{P}$,

$$\alpha_\omega \left(p\left(\psi_1\left(\omega\right), \psi_2\left(\omega\right), \ldots\right)\right) = p\left(\varphi_1\left(\omega\right), \varphi_2\left(\omega\right), \ldots\right)$$

defines a unital algebra homomorphism α_ω from

$$\left\{p\left(\psi_1\left(\omega\right), \psi_2\left(\omega\right), \ldots\right) : p \in \mathcal{P}\right\}$$

to

$$\left\{p\left(\varphi_1\left(\omega\right), \varphi_2\left(\omega\right), \ldots\right) : p \in \mathcal{P}\right\}$$

with $\|\alpha_\omega\| \leq \|\sigma_1\| \leq \|\rho\|^3$. The Claim 8.3 shows that, for each $\omega \in \Omega$, α_ω is d_K-d_K uniformly continuous on $\mathcal{P}_1$. Hence α_ω uniquely extends to a unital algebra homomorphism from $\mathcal{M}_\omega$ that is ultrastrong-ultrastrong continuous on the unit ball of $\mathcal{M}_\omega^{sa}$. It follows that if $\psi \in L^\infty\left(\mu, B\left(K\right)\right)$ is in $\mathcal{M}$ and $\sigma_1\left(\psi\right) = \varphi \in L^\infty\left(\mu, B\left(K\right)\right)$, then, for every $\omega \in \Omega$,

$$\varphi\left(\omega\right) = \alpha_\omega\left(\psi\left(\omega\right)\right).$$

It follows that

$$\|\sigma_1\|_{cb} \leq \sup_{\omega \in \Omega} \|\alpha_\omega\|_{cb}.$$

If the factor $\mathcal{M}_\omega$ has type I , then it it hyperfinite, so $\hat{d}_*\left(\mathcal{M}_\omega\right) \leq 2$. If $\mathcal{M}_\omega$ has type II_∞ or type III, then U. Haagerup's result [**2**] implies $\hat{d}_*\left(\mathcal{M}_\omega\right) \leq 3$. If the factor $\mathcal{M}_\omega$ has type II_1, then $\hat{d}_*\left(\mathcal{M}_\omega\right) \leq \hat{d}_{tr}\left(\mathcal{A}\right)$ by Definition 7. Hence,

$$\sup_{\omega\in\Omega} \hat{d}_*\left(\mathcal{M}_\omega\right) \leq \max\left(3, \hat{d}_{tr}\left(\mathcal{A}\right)\right).$$

Therefore

$$\left\|\sigma_1\right\|_{cb} \leq \sup_{\omega\in\Omega} \max\left(3, \hat{d}_{tr}\left(\mathcal{A}\right)\right) \left\|\alpha_w\right\|^{\max\left(3,\hat{d}_{tr}(\mathcal{A})\right)} \leq \max\left(3, \hat{d}_{tr}\left(\mathcal{A}\right)\right) \left\|\rho\right\|^{3\max\left(3,\hat{d}_{tr}(\mathcal{A})\right)},$$

and

$$\left\|\rho\right\|_{cb} = \left\|\sigma\right\|_{cb} \leq \left\|\sigma_1\right\|_{cb} \left\|\rho\right\|^2 \leq 3\max\left(3, \hat{d}_{tr}\left(\mathcal{A}\right)\right) \left\|\rho\right\|^{2+3\max\left(3,\hat{d}_{tr}(\mathcal{A})\right)}.$$

It follows that

$$d\left(\mathcal{A}\right) \leq \hat{d}\left(\mathcal{A}\right) \leq 2 + 3\max\left(3, \hat{d}_{tr}\left(\mathcal{A}\right)\right).$$

□

Erik Christensen [**1**] proved that if $\mathcal{M}$ is a II_1 factor with property Γ, then $d\left(\mathcal{M}\right) = 3$ and $\kappa\left(\mathcal{M}\right) = 1$, which means $\hat{d}\left(\mathcal{M}\right) \leq 3$. We will not name or try to give an internal description of the class of C*-algebras in the following Corollary, but we note that it clearly contains the class of weakly approximately divisible C*-algebras defined in [**4**].

Corollary 9. *Suppose $\mathcal{A}$ is a separable unital C*-algebra with that property that $\pi_\tau\left(\mathcal{A}\right)''$ has property Γ for every factor tracial state τ on $\mathcal{A}$. Then*

$$d\left(\mathcal{A}\right) \leq \hat{d}\left(\mathcal{A}\right) \leq 11.$$

Note that the following corollary relates $d\left(C^*\left(\mathbb{F}_n\right)\right)$ and $\hat{d}_*\left(\mathcal{L}_{\mathbb{F}_n}\right)$ for each integer $n \geq 2$.

Corollary 10. *If $\mathcal{A}$ is a separable unital C*-algebra with a unique tracial state τ, then*

$$d\left(\mathcal{M}_\tau\left(\mathcal{A}\right)\right) \leq \hat{d}_*\left(\mathcal{M}_\tau\left(\mathcal{A}\right)\right) \leq d\left(\mathcal{A}\right) \leq 2 + 3\max\left(3, \hat{d}_*\left(\mathcal{M}_\tau\left(\mathcal{A}\right)\right)\right).$$

We now apply our results to certain crossed products.

Suppose $n \geq 2$ is a positive integer and let $\mathbb{G}_n$ denote the group generated by the $n \times n$ diagonal unitary matrices and the permutation matrices. We define an action α on $C_r^*\left(\mathbb{F}_n\right)$ with standard unitary generators $u_1, \ldots, u_n$, so that if $g = DV_\sigma$ with $D = diag\left(\lambda_1, \ldots, \lambda_n\right)$ and V_σ is the permutation matrix corresponding to $\sigma \in S_n$, we let $\alpha\left(g\right)$ be the automorphism of $C_r^*\left(\mathbb{F}_n\right)$ that sends u_k to $\lambda_k u_{\sigma(k)}$.

Corollary 11. *Suppose $n \geq 2$ and H is an abelian subgroup of $\mathbb{G}_n$ that is not torsion (i.e., H contains at least one element of infinite order). Then*

$$\hat{d}_{tr}\left(C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H\right) \leq 3,$$

and

$$d\left(C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H\right) \leq 11.$$

PROOF. Suppose $h \in H$ has infinite order. Then $h^{n!} \neq I_n$ (the $n \times n$ identity matrix, which is the identity element of $\mathbb{G}_n$). However, $h^{n!}$ must be a diagonal matrix $D = diag\left(\lambda_1, \ldots, \lambda_n\right)$. Moreover, $h^{n!}$ has infinite order, so λ_{j_0} must be irrational for some $1 \leq j_0 \leq n$. We also know that there is an increasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $D^{m_k} \to I_n$, i.e., $\lim_{k\to\infty} \lambda_j^{m_k} = 1$ for $1 \leq j \leq n$.

Let $W = W_{h^{n!}}$ be the unitary in $C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H$ corresponding to $h^{n!}$, so that, for every $A \in C_r^*\left(\mathbb{F}_n\right)$,

$$WAW^* = \alpha\left(h^{n!}\right)(A).$$

Since

$$\left\|W^{m_k} u_j \left(W^{m_k}\right)^* - u_j\right\| = \left|\lambda_j^{m_k} - 1\right| \left\|u_j\right\| \to 0$$

and, since H is abelian $WW_g = W_g W$ for every $g \in H$. Hence

$$\left\|W^{m_k} T - T W^{m_k}\right\| \to 0$$

for every $T \in C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H$.

Next suppose τ is a tracial state on $C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H$ and m is a positive integer. Since

$$W^m u_{j_0} = \lambda_{j_0}^m u_{j_0} W^m,$$

we conclude that $\tau\left(W^m\right) = 0$. It follows that $\mathcal{M}_\tau\left(C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H\right)$ is a von Neumann algebra with property Γ, so by [**1**],

$$\hat{d}_*\left(\mathcal{M}_\tau\left(C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H\right)''\right) \leq 3.$$

Hence

$$\hat{d}_{tr}\left(C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H\right) \leq 3,$$

and

$$d\left(C_r^*\left(\mathbb{F}_n\right) \rtimes_\alpha H\right) \leq 11.$$

□

REMARK 12. We conclude by reminding the reader of the equivalent formulations of Kadison's similarity problem (see, e.g., [**14**]) so, for example, showing $d(\mathcal{A}) < \infty$ implies that if $\pi : \mathcal{A} \to B(H)$ is a unital $*$-homomorphism and $T \in B(H)$ and $T(H)$ is invariant for every operator in $\pi(\mathcal{A})$, then there is an $S \in \pi(\mathcal{A})'$ such that $S(H) = T(H)$. Moreover, there is a $K \geq 1$ such that, for every $W \in B(H)$,

$$dist\left(W, \pi(\mathcal{A})'\right) \leq K \sup\left\{\left\|\pi(U) W - W\pi(U)\right\| : U \in \mathcal{A}, U \text{ unitary}\right\}.$$

References

[1] E. Christensen, *Finite von Neumann algebra factors with property* Γ, J. Funct. Anal. 186(2001), 366-380.
[2] U. Haagerup, Solution of the similarity problem for cyclic representations of C*-algebras, Ann. Math. 118 (1983), 215-240.
[3] D. Hadwin, Dilations and Hahn decompositions for linear maps, Canad. J. Math. 33 (1981) 826–839.
[4] D. Hadwin and Weihua Li, The similarity degree of some C*-algebras, arXiv:1210.6015, 2012.
[5] D. Hadwin and Xiujuan Ma, A note on free products, Oper. Matrices 2 (2008) 53–65.
[6] M. Johanesová and W. Winter, *The similarity problem for $\mathcal{Z}$-stable C*-algebras*, Preprint 1104.2067, 2011.
[7] R. Kadison, *On the orthogonalization of operator representations*, Amer. J. Math. 77 (1955), 600-622.
[8] W. Li, *The similarity degree of approximately divisible C*-algebras*, Preprint, 2012, to appear, Operators and Matrices.
[9] F. Perera, A. Toms, S. White, W. Winter, The Cuntz semigroup and stability of close C*-algebras, arXiv:1210.4533, 2012.
[10] G. Pisier, Grothendieck's theorem for noncommutative C*-algebras, with an appendix on Grothendieck's constants. J. Funct. Anal. 29 (1978) 397–415.
[11] G. Pisier, *The similarity degree of an operator algebra*, Algebra i Analiz 10(1998), 132–186; translation in St. Petersburg Math. J. 10 (1999), 103–146.
[12] G. Pisier, *Remarks on the similarity degree of an operator algebra*, Internat. J. Math. 12 (2001), 403-414.
[13] G. Pisier, *Similarity problems and length*, Taiwanese J. Math. 5 (2001), 1-17.
[14] G. Pisier, *Similarity problems and completely bounded maps*, Second, expanded edition, Lecture Notes in Math., vol. 1618, Springer-Verlag, Berlin 2001.
[15] G. Pisier, *A similarity degree characterization of nuclear C*-algebras*, Publ. Res. Inst. Math. Sci. 42 (2006), No. 3, 691-704.
[16] F. Pop, *The similarity problem for tensor products of certain C*-algebras*, Bull. Austral. Math. Soc. 70 (2004), 385-389.
[17] Ringrose, J. R., Linear mappings between operator algebras. Symposia Mathematica, Vol. XX (Convegno sulle Algebre iC e loro Applicazioni in Fisica Teorica, Convegno sulla Teoria degli Operatori Indice e Teoria K, INDAM, Roma, 1975), pp. 297–315. Academic Press, London, 1976.
[18] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. 40 (1981) 127–150.

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